

Algebraic models of Merge and phases for tensor networks

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Overview

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models of
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The Minimalist Program

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- Introduced by Chomsky in 1993 [3]
- “What is language? Why does it have the properties it has?”
- Assumptions
 - Language is a form of cognition (there exists a language faculty)
 - Language is a computational system with invariant principles and parameters
 - Language acquisition consists of acquiring a lexicon and setting parameter values
 - Language generates an infinite set of expressions given as a sound-meaning pair (π, λ)
- In SMT, the computational system has only one basic operation, *Merge* [2]

Merge

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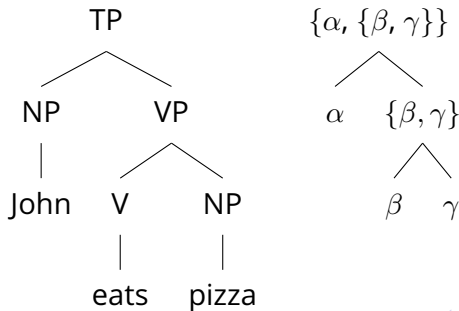
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Simply assigns two arguments α and β to an unordered set

$$\mathfrak{M}(\alpha, \beta) := \{\alpha, \beta\}.$$

- **External Merge (EM):** α and β are separate syntactic objects
- **Internal Merge (IM):** either α or β is part of the other



Syntactic Objects

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Given a lexicon $\mathcal{SO}_0$, we define the set of syntactic objects $\mathcal{SO}$ as the free, non-associative, commutative magma over $\mathcal{SO}_0$ with the merge operation $\mathfrak{M}$,

$$\mathcal{SO} := \text{Magma}_{na,c}(\mathcal{SO}_0, \mathfrak{M}).$$

$\mathcal{SO}$ can also be identified with the set $\mathfrak{T}_{\mathcal{SO}}$. [6]

Phases

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- Sub-computations that build syntactic objects up to a certain level
- Disagreements about what constitutes a phase. [4]
 - Every CP (complementizer phrase) and some vPs are phases
 - A specified set of phrases are phases, depending on the language
 - Every phrase is a phase

Phase Impenetrability

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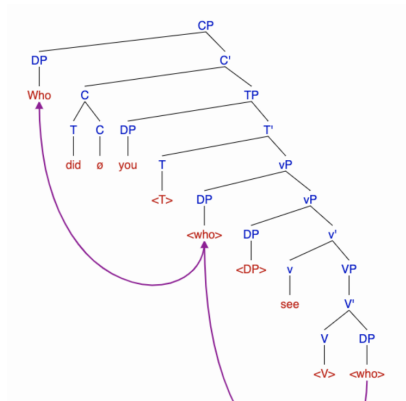
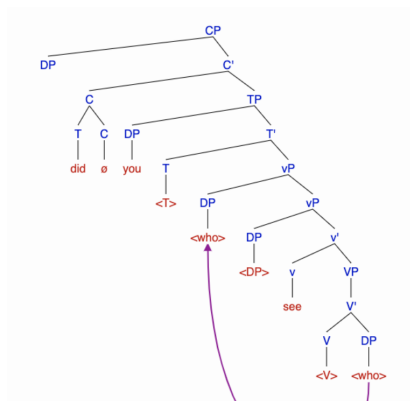
Phase Impenetrability Condition (PIC): Movement of a syntactic object out of a phase is permitted only if the object has moved to the left edge of the phase.

Consider the sentence: “Who did you see?”

Phase Impenetrability

Phase Impenetrability Condition (PIC): Movement of a syntactic object out of a phase is permitted only if the object has moved to the left edge of the phase.

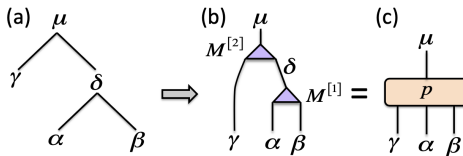
Consider the sentence: "Who did you see?"



Renormalization Models

Recent proposals by Pestum, Vlassopoulos, Gallego, Orus: [8, 7, 5]

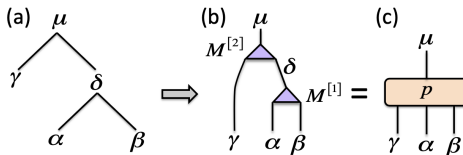
- Use tensor networks to model hierarchical structure
- Each Merge is a small tensor map
- MERA adds disentangler before course-graining



Renormalization Models

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- Use tensor networks to model hierarchical structure
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Two problems:

- Associative
- No phase structure

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Construct algebraic model of Merge that is compatible with tensor networks, while maintaining non-associativity and phase structure.

Constructions

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- Jordan-Banach operator algebra
- Grossman-Larson algebra of trees

Special Jordan Algebras

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- Given an associative algebra A over field of characteristic $\neq 2$.
- Define Jordan product

$$\alpha \circ \beta := \frac{1}{2}(\alpha\beta + \beta\alpha)$$

- Satisfies:
 - $x \circ y = y \circ x$
 - $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$

Sketch of Construction

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- Let $J_0 = T(\mathcal{SO}_0)$ be the free non-associative tensor algebra over the set of lexical items.
- Quotient by the ideal generated by the Jordan relations to get Jordan algebra $J = J_0/I$.
- Inherit norm from universal enveloping C^* -algebra $C^*(J)$ to obtain Jordan-Banach algebra. [1] There exists a maximal Jordan ideal I_J such that J/I_J is a JB-algebra.
- The universal representation is

$$\Pi : J \rightarrow \bigoplus_{\phi \in S(J)} B(H_\phi)_{sa}$$

The Kernel

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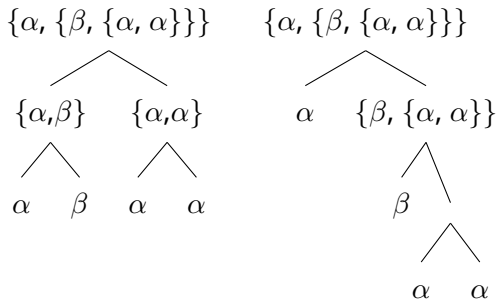
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The kernel of the universal representation Π is the maximal Jordan ideal I_J .



Sketch of Grossman–Larson Construction

- For each $w \in \mathcal{SO}_0$ assign polynomial $a_w(x) \in \mathbb{R}[x]$.
- Define leaf operator by

$$E_w(x) = a_w(x) \frac{d}{dx},$$

- Symmetrize over planar embeddings:

$$D_t = \frac{1}{k!} \sum_{\sigma \in S_k} D_{t_{\sigma(1)}} \circ \cdots \circ D_{t_{\sigma(k)}}$$

- For fixed Gaussian $\varphi(x) = e^{-(x-\mu)^2/2}$, our operator acts by

$$D_t \varphi(x) = Q_t(x) \varphi(x),$$

- When phase head is encountered, project operator onto corresponding subspace.

Example

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- Use three leaves α, β, γ with $E_\alpha(x) = x \partial_x$, $E_\beta(x) = \partial_x$, $E_\gamma(x) = x^2 \partial_x$.
- Probe: Gaussian $\varphi(x) = e^{-x^2/2}$ so $\varphi' = -x\varphi$, etc.
- Compare left- vs right-nested trees:

$$t_L = ((\alpha \beta) \gamma), \quad t_R = (\alpha (\beta \gamma)).$$

We obtain

$$D_{t_L} \varphi(x) = (x^5 - 3x^3 + x) \varphi(x), \quad D_{t_R} \varphi(x) = (x^5 - 2x^3) \varphi(x).$$

Truncating at degree $K = 5$ (basis $1, x, \dots, x^5$) gives coefficient vectors:

$$Q_{t_L} \leftrightarrow [0, 1, 0, -3, 0, 1]^\top, \quad Q_{t_R} \leftrightarrow [0, 0, 0, -2, 0, 1]^\top.$$

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