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Positroid stratifications of positive symplectic Grassmannians

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Overview

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- $\text{Gr}(k, n)$ is the space of all k -dimensional subspaces of $\mathbb{R}^n$.
- Or $\{k \times n \text{ matrices of rank } k\} / \text{row operations}$

Plücker coordinates

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- Let A be a $k \times n$ matrix. For each $I \subseteq [n]$ with $|I| = k$, let $\Delta_I(A)$ be the determinant of the $k \times k$ submatrix of A with columns indexed by I .
- The Plücker coordinates of a point in $\text{Gr}(k, n)$ are the collection of all $\Delta_I(A)$ for all $I \subseteq [n]$ with $|I| = k$.

Example

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $\Delta_{12}(A) = \det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = 1 \cdot 5 - 2 \cdot 4 = -3$.

Totally nonnegative Grassmannian

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The totally nonnegative Grassmannian $\text{Gr}_{\geq 0}(k, n)$ is the subset of $\text{Gr}(k, n)$ where all Plücker coordinates are nonnegative.

Postnikov decomposes $\text{Gr}_{\geq 0}(k, n)$

For a point $X \in \text{Gr}_{\geq 0}(k, n)$, record exactly which Plücker coordinates are positive:

$$\mathcal{M}(X) = \left\{ I \in \binom{[n]}{k} : \Delta_I(X) > 0 \right\}.$$

The possible collections $\mathcal{M}$ are **positroids**, and each determines a cell

$$\mathcal{S}_{\mathcal{M}}^{>0} = \{X : \Delta_I(X) > 0 \iff I \in \mathcal{M}\}.$$

Positroid decomposition

$$\text{Gr}_{\geq 0}(k, n) = \bigsqcup_{\mathcal{M}} \mathcal{S}_{\mathcal{M}}^{>0}.$$

Each cell is an open ball. Cells are indexed by positroids, decorated permutations, Grassmann necklaces, or reduced plabic graphs modulo moves.

Plabic graphs

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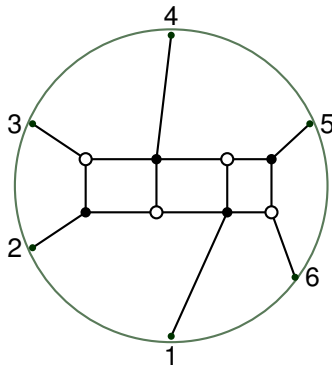
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- **Plabic** means *planar bicolored*.
- The graph is embedded in a disk
- Boundary vertices are labeled $1, \dots, n$
- Internal vertices are colored black or white

The embedding divides the disk into **faces**.



A perfect orientation turns the graph into a network

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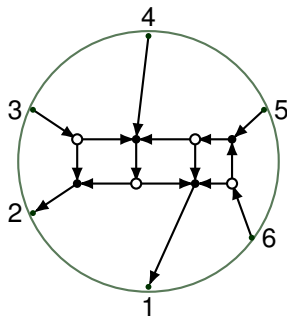
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In a **perfect orientation**:

- there are $n - k$ sources and k sinks
- every black vertex has exactly one outgoing edge
- every white vertex has exactly one incoming edge

A boundary edge points away from a **source** and toward a **sink**.

Face weights give eight independent positive parameters

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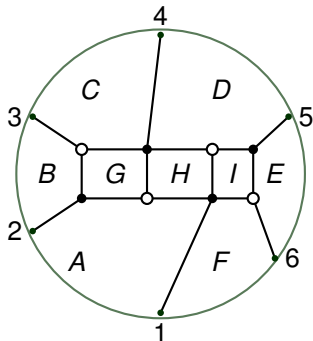
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Assign a positive weight to every face:

$$A, B, \dots, I > 0.$$

One overall rescaling is redundant, so we normalize

$$ABCDEFGHI = 1.$$

Thus the graph has

$$9 - 1 = 8 = \dim \operatorname{Gr}(2, 6)$$

independent parameters.

Directed paths build the boundary-measurement matrix

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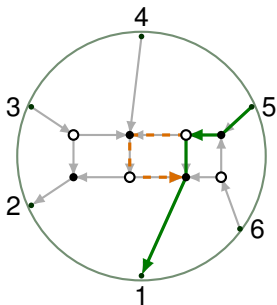
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Each entry sums the weights of all directed paths from a source to a sink.

$$M = \begin{pmatrix} 1 & 0 & -e & -f & -g & -h \\ 0 & 1 & a & b & c & d \end{pmatrix}.$$

From source 5 to sink 1 there are two paths, with face weights

$$EFI \text{ and } EFHI.$$

Therefore

$$g = EFI + EFHI = EFI(1 + H).$$

A reduced graph is a positive chart for one cell

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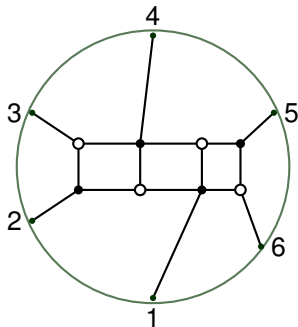
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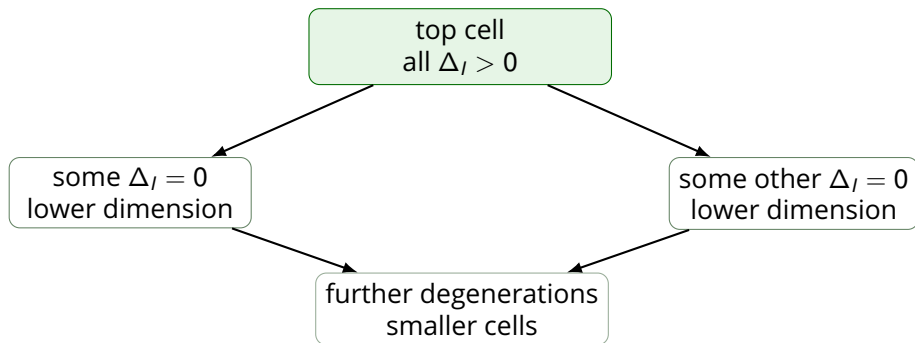
A plabic graph is **reduced** when its trips have

- no closed trips
- no essential self-intersections
- no bad double crossings

Our graph is reduced. Its eight independent positive face weights give a subtraction-free parametrization of the **top positroid cell**:

$$\Delta_I > 0 \quad \text{for every } I \in \binom{[6]}{2}.$$

Lower-dimensional cells form the boundary of the top cell



- A cell's dimension equals the number of independent parameters in a reduced chart.
- Degenerating parameters adds zeros and moves to lower-dimensional cells.
- Closure containment defines the positroid poset.

The symplectic Grassmannian

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Let Ω be a nondegenerate skew-symmetric $2n \times 2n$ matrix, defining

$$\omega(u, v) = u^T \Omega v.$$

The **symplectic Grassmannian** is

$$\mathrm{SpGr}(k, 2n) = \{V \in \mathrm{Gr}(k, 2n) : \omega(u, v) = 0 \text{ for every } u, v \in V\}.$$

In other words, it parametrizes k -dimensional **isotropic subspaces**.
If the rows of a full-rank matrix M span V , then equivalently

$$M\Omega M^T = 0.$$

Its totally nonnegative part is

$$\mathrm{SpGr}_{\geq 0}(k, 2n) = \mathrm{SpGr}(k, 2n) \cap \mathrm{Gr}_{\geq 0}(k, 2n).$$

Coordinate models of a symplectic form

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Every nondegenerate alternating form on $\mathbb{R}^{2n}$ is equivalent after a change of basis, but the chosen basis matters for Plücker positivity.

Anti-diagonal form

$$(\Omega_{\text{ad}})_{ij} = \begin{cases} (-1)^{i+1}, & i + j = 2n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Block and rotational forms

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad R_n = \begin{pmatrix} 0 & D_n \\ -D_n & 0 \end{pmatrix},$$

$$D_n = \text{diag}(1, -1, 1, -1, \dots).$$

If $\Omega' = S^T \Omega S$, then S identifies their isotropic varieties. However, S can permute or change signs of Plücker coordinates, so it need not identify their intersections with $\text{Gr}_{\geq 0}$.

Computing the symplectic slice cell by cell

For $\mathrm{SpGr}(2, 6)$, isotropy is the single Plücker equation

$$F = \Delta_{16} - \Delta_{25} + \Delta_{34} = 0.$$

I imposed this equation cell by cell:

- 1 enumerate the decorated permutations indexing the positroid cells of $\mathrm{Gr}_{\geq 0}(2, 6)$;
- 2 compute a reduced plabic graph for every cell and choose a perfect orientation;
- 3 assign its independent positive parameters $x_1, \dots, x_d$ and use boundary measurement to compute every $\Delta_{ij}(x)$;
- 4 substitute them into F to obtain a cell-specific equation $F_C(x) = 0$.

Thus the symplectic piece of an ordinary positroid cell C is

$$C^\omega = \{x \in \mathbb{R}_{>0}^d : F_C(x) = 0\}.$$

Each cell has one of three outcomes

Let $d = \dim C$ be the number of independent positive parameters in the plabic-graph parametrization. After simplifying $F_C(x)$ on $\mathbb{R}_{>0}^d$, I assigned the **symplectic dimension**

$$d_\omega(C) = \begin{cases} d, & F_C \equiv 0, \\ d-1, & F_C = 0 \text{ cuts out a nonempty hypersurface,} \\ -1, & F_C = 0 \text{ has no positive solution.} \end{cases}$$

- $F_C \equiv 0$: the whole positroid cell is isotropic.
- One genuine equation: the symplectic condition lowers the dimension by one.
- No positive solution: the positroid cell does not meet $\mathrm{SpGr}_{\geq 0}(2, 6)$.

These data supplied the nodes of the tentative symplectic poset.

Inheriting the ordinary cover relations

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In the ordinary positroid poset,

$$C_p \longrightarrow C_q \quad \Longleftrightarrow \quad C_q \subseteq \overline{C_p} \text{ is a cover relation.}$$

To form a first candidate for the symplectic poset, I

- 1 kept the nonempty slices C^ω ;
- 2 inherited the ordinary parent-child relations;
- 3 labeled every node by its computed dimension $d_\omega(C)$.

For a CW cell decomposition, a boundary cell must have strictly smaller dimension:

$$C_q^\omega \subseteq \overline{C_p^\omega} \implies d_\omega(C_q) < d_\omega(C_p).$$

Checking this inequality exposed the problematic relations.

Six inherited covers fail the dimension test

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parent	child	symplectic dimensions
$C37$	$C56$	$4 \rightarrow 4$
$C55$	$C59$	$4 \rightarrow 4$
$C63$	$C90$	$3 \rightarrow 3$
$C71$	$C96$	$3 \rightarrow 3$
$C72$	$C87$	$3 \rightarrow 3$
$C76$	$C97$	$3 \rightarrow 3$

In every pair, the symplectic equation lowers the parent's dimension but does not lower the child's. The inherited ordinary cover therefore becomes a same-dimensional relation, so the *whole* child cannot be a boundary cell of the parent.

The $C63 \rightarrow C90$ attachment

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On the parent $C63$,

$$F_C = -x_4 + x_2 = 0 \implies x_2 = x_4.$$

Using

$$\Delta_{26} = x_3 x_4, \quad \Delta_{36} = x_2 x_3 x_4,$$

$$\Delta_{46} = x_1 x_2, \quad \Delta_{56} = x_1 x_2 x_4,$$

every point of $\overline{C63}^\omega$ satisfies

$$\Delta_{26} \Delta_{56} = \Delta_{36} \Delta_{46}.$$

The parent attaches along a codimension-one subset of the child, not along the entire child.

On the child $C90$,

$$\Delta_{26} = 1, \quad \Delta_{36} = u_3,$$

$$\Delta_{46} = u_2 u_3, \quad \Delta_{56} = u_1 u_2 u_3.$$

Substitution gives

$$u_1 u_2 u_3 = u_2 u_3^2 \implies u_1 = u_3.$$

Therefore

$$\overline{C63}^\omega \cap C90 \subsetneq C90.$$

Picture

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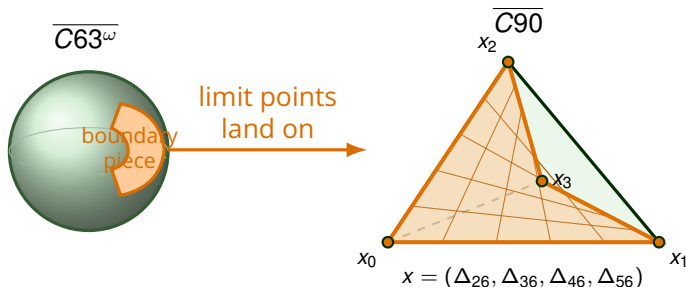
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$$\overline{C63^\omega} \cap C90 \subseteq C90 \cap \{\Delta_{26}\Delta_{56} = \Delta_{36}\Delta_{46}\} \subsetneq C90.$$



Each problematic cover imposes an extra equation

pair	equation on the parent closure	equation inside the child
$C37 \rightarrow C56$	$\Delta_{13}\Delta_{36} = \Delta_{23}\Delta_{35}$	$u_1 = u_3$
$C55 \rightarrow C59$	$\Delta_{12}\Delta_{15} = \Delta_{13}\Delta_{14}$	$u_2 = u_4$
$C63 \rightarrow C90$	$\Delta_{26}\Delta_{56} = \Delta_{36}\Delta_{46}$	$u_1 = u_3$
$C71 \rightarrow C96$	$\Delta_{13}\Delta_{36} = \Delta_{23}\Delta_{35}$	$u_1 = u_3$
$C72 \rightarrow C87$	$\Delta_{12}\Delta_{15} = \Delta_{13}\Delta_{14}$	$u_1 = u_3$
$C76 \rightarrow C97$	$\Delta_{14}\Delta_{46} = \Delta_{24}\Delta_{45}$	$u_1 = u_3$

- In all six cases, the parent closure meets only a proper subset of the child.
- For the four 3-dimensional children, the child closure is combinatorially a tetrahedron, while the attaching surface has closure combinatorially a square.

Therefore the ordinary positroid stratification restricted to $\text{SpGr}_{\geq 0}(2, 6)$ is not itself a CW cell decomposition.

Positive flag varieties

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Define the rotational nonnegative half-flag space

$$\mathcal{F}_3^{R, \geq 0} = \left\{ V_1 \subset V_2 \subset V_3 \subset \mathbb{R}^6 \mid \begin{array}{l} \dim V_i = i, \quad V_i \in \text{Gr}_{\geq 0}(i, 6), \\ V_3 \text{ is } R_3\text{-Lagrangian} \end{array} \right\}.$$

If R_3 vanishes on V_3 , it automatically vanishes on V_1 and V_2 . Thus the symplectic equations occur only on the top 3-plane.

The ambient space $\text{Fl}_{\geq 0}(1, 2, 3; 6)$ is decomposed into positive Richardson cells $\mathcal{R}_{u, w}^{\geq 0}$, indexed by Bruhat intervals $u \leq w$.

How to obtain the reduced word

In S_6 , write $s_i = (i, i + 1)$. The open cell of $\text{Fl}(1, 2, 3; 6)$ is represented by the longest minimal coset representative

$$w_0^J = [6, 5, 4, 1, 2, 3], \quad J = \{4, 5\}.$$

Bubble-sorting to this permutation records the adjacent swaps:

$$\begin{aligned} 123456 &\xrightarrow{345} 124563 \xrightarrow{234} 145623 \\ &\xrightarrow{123} 456123 \xrightarrow{121} 654123. \end{aligned}$$

Hence

$$\underbrace{345 \ 234 \ 123}_{w_{\text{top}} = [4, 5, 6, 1, 2, 3] \text{ move the top 3-plane}} \quad \underbrace{121}_{w_{\text{fiber}} = [3, 2, 1, 4, 5, 6] \text{ flag inside } V_3}.$$

The two pieces have 9 and 3 inversions, so the word has length $12 = \dim \text{Fl}(1, 2, 3; 6)$ and is reduced.

Chevalley generators turn the word into coordinates

For SL_6 , the lower simple-root subgroup is

$$y_i(t) = I_6 + tE_{i+1,i}, \quad t > 0.$$

Replace the successive letters of the reduced word by these matrices:

$$g(\mathbf{t}) = y_3(t_1)y_4(t_2)y_5(t_3)y_2(t_4)y_3(t_5)y_4(t_6) \\ \cdot y_1(t_7)y_2(t_8)y_3(t_9)y_1(t_{10})y_2(t_{11})y_1(t_{12}).$$

The resulting flag is read from the first columns:

$$V_j = \text{span}(g(\mathbf{t})\mathbf{e}_1, \dots, g(\mathbf{t})\mathbf{e}_j), \quad j = 1, 2, 3.$$

Open Marsh–Rietsch chart

As $(t_1, \dots, t_{12})$ ranges over $\mathbb{R}_{>0}^{12}$, this parametrizes the open positive Richardson cell of $\text{Fl}(1, 2, 3; 6)$.

Marsh–Rietsch coordinates for boundary cells

Fix a reduced word $w = s_{i_1} \cdots s_{i_m}$ and $v \leq w$. Its **positive distinguished subexpression** is found from right to left:

$$v_{(m)} = v, \quad v_{(j-1)} = \begin{cases} v_{(j)} s_{i_j}, & v_{(j)} s_{i_j} < v_{(j)}, \\ v_{(j)}, & v_{(j)} s_{i_j} > v_{(j)}. \end{cases}$$

Then replace each letter by

$$g_j = \begin{cases} \dot{s}_{i_j}, & \text{the letter is selected,} \\ y_{i_j}(t_j), & t_j > 0, \text{ the letter is omitted,} \end{cases} \quad g = g_1 \cdots g_m,$$

where $\dot{s}_i$ has the 2×2 block $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in rows and columns $i, i+1$.

- The omitted letters supply exactly $\ell(w) - \ell(v)$ positive parameters.
- For the open cell $v = e$, every letter is omitted, so every factor is $y_i(t)$.

The last three factors only move the lower flag

Split the word and the product along the projection to $\text{Gr}(3, 6)$:

$$\underbrace{345\,234\,123}_{g_{\text{top}}} \underbrace{121}_{g_{\text{fiber}}}, \quad g = g_{\text{top}} g_{\text{fiber}}.$$

If C is the first three columns of g , direct multiplication gives

$$C = C_0 H, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ t_{10} + t_{12} & 1 & 0 \\ t_{11} t_{12} & t_{11} & 1 \end{pmatrix},$$

where C_0 depends only on $t_1, \dots, t_9$. Since H is invertible,

$$\text{span}(C) = \text{span}(C_0), \quad C^T R_3 C = H^T (C_0^T R_3 C_0) H.$$

Therefore the rotational Lagrangian equations involve only the first nine parameters. t_{10}, t_{11}, t_{12} refine the flag inside the fixed V_3 .

The rotational condition gives three equations

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Direct multiplication of $C_0^T R_3 C_0 = 0$ gives

$$0 = t_1 t_4 + t_1 t_8 + t_5 t_8 - t_2 t_5 t_7 t_8,$$

$$0 = t_1 + t_5 + t_9 - t_2 t_5 t_7 - t_2 t_7 t_9 + t_3 t_4 t_6 t_7 t_9 - t_6 t_7 t_9,$$

$$0 = -t_2 t_5 - t_2 t_9 + t_3 t_4 t_6 t_9 + t_3 t_6 t_8 t_9 - t_6 t_9.$$

Take

$$t_3, t_4, t_5, t_6, t_8, t_9$$

as free coordinates. The equations then solve uniquely for t_1, t_2, t_7 , and all three solutions are positive precisely when

$$t_3 t_4 > 1.$$

A positive reparametrization absorbs the inequality

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Use arbitrary $a, b, c, d, e, r > 0$:

$$\begin{aligned}t_4 &= a, & t_5 &= b, & t_6 &= c, \\t_8 &= d, & t_9 &= e, & t_3 &= \frac{1+r}{a}, \\ \Lambda &= ab + (1+r)e(a+d).\end{aligned}$$

Then $t_3 t_4 - 1 = r > 0$.

The remaining top-plane coordinates are

$$\begin{aligned}t_1 &= \frac{ab(b+e)r}{\Lambda}, \\t_2 &= \frac{ce(ra + (1+r)d)}{a(b+e)}, \\t_7 &= \frac{a(b+e)(a(b+e) + de)}{cde\Lambda}.\end{aligned}$$

All three are positive for every
 $(a, b, c, d, e, r) \in \mathbb{R}_{>0}^6$.

Finally set $t_{10} = u, t_{11} = v, t_{12} = w$ with $u, v, w > 0$.

The open rotational flag locus is a 9-cell

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12 positive MR parameters
for the open flag cell

3 independent equations
from $C_0^T R_3 C_0 = 0$

9 free positive parameters
($a, b, c, d, e, r, u, v, w$)

The first six parameters determine the rotational positive Lagrangian 3-plane. The last three determine a positive complete flag inside it.

Proved for the open cell

$$\mathcal{F}_3^{R, >0} \cong \text{LG}_{>0}^R(3, 6) \times \mathbb{R}_{>0}^3 \cong \mathbb{R}_{>0}^9.$$

This matches the dimension count $\dim \text{LG}(3, 6) + \dim \text{Fl}(1, 2; 3) = 6 + 3 = 9$.

Extending the factorization to the boundary is the next step

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Let $C_{\mathbf{P}}$ be an ordinary positive flag cell, with top positroid cell $\Pi_{P_3}^{>0} \subset \text{Gr}_{\geq 0}(3, 6)$. The open-cell calculation suggests the cellwise factorization

$$C_{\mathbf{P}}^R \stackrel{?}{\cong} (\Pi_{P_3}^{>0} \cap \text{LG}_{\geq 0}^R(3, 6)) \times \mathbb{R}_{>0}^{r(\mathbf{P})}, \quad r(\mathbf{P}) = \dim C_{\mathbf{P}} - \dim \Pi_{P_3}^{>0}.$$

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A point is a pair

$$L \subset V \subset \mathbb{R}^n, \quad \dim L = 1, \quad \dim V = 3,$$

with both spaces nonnegative.

We can organize the problem by first remembering only the line:

$$(L, V) \longmapsto L.$$

After normalizing its coordinates, a nonnegative line is a point in a simplex:

$$x_i \geq 0, \quad \sum_i x_i = 1, \quad \text{Gr}_{\geq 0}(1, n) \cong \Delta_{n-1}.$$

The line tells us which case we are in

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The **support** I records the positions where the line is nonzero.

Consecutive support

$$I = \{1, 2\} \longleftrightarrow (*, *, 0, \dots)$$

Separated support

$$I = \{1, 3\} \longleftrightarrow (*, 0, *, \dots)$$

Cases with consecutive support and $|I| \leq 3$ are already understood. The first new case is

$$I = \{1, 3\}.$$

After fixing the line, only the 3-plane remains

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Once the support I is fixed, a positive rescaling lets us replace the line by

$$\text{span}(\mathbf{1}_I), \quad (\mathbf{1}_I)_j = \begin{cases} 1, & j \in I, \\ 0, & j \notin I. \end{cases}$$

The problem is then to understand the nonnegative 3-planes containing this one fixed vector:

$$\text{Gr}_I^{\geq 0}(3, n) = \{V \in \text{Gr}_{\geq 0}(3, n) : \mathbf{1}_I \in V\}.$$

So, after fixing I , there are two choices: where the line sits in its simplex face, and which 3-plane contains $\mathbf{1}_I$.

What changes for $I = \{1, 3\}$?

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Here the fixed vector is

$$\mathbf{1}_{\{1,3\}} = (1, 0, 1, 0, \dots).$$

- some Plücker coordinates are forced to be zero
- the top positroid cell is ruled out
- the whole problem takes place on the positroid boundary

What I want to understand

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The immediate goal is to describe

$$\mathrm{Gr}_{\{1,3\}}^{\geq 0}(3, n).$$

Some concrete questions:

- Which positroid cells can contain these 3-planes?
- What does the intersection look like inside each surviving cell?
- Do those pieces already form cells, or do they need to be split further?

Understanding this first separated support should point toward the more general problem for arbitrary $I \subseteq [n]$.

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Feedback

